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Report on the thesis manuscript of Jacek Gatlik "Kink Dynamics in the sine-Gordon Model: Interaction with Inhomogeneities"

Jacek Gatlik has written his doctoral thesis under the supervision of Professors Tomasz Dobrowolski and Panayotis Kevrekides. He had already four publications in a different field of physics and this is a great accomplishment.

The topic of the thesis is the analysis in different contexts of the dynamics of kinks in a sine-Gordon equation with a perturbation of the spatial term. Gatlik analyzes these kink solutions using the numerical solutions of the partial differential equations and "collective variable" models based on the center of the kink or its width. These studies are important for the theory of nonlinear waves and for applications. In particular, the driven damped sine-Gordon equation is an excellent model of Josephson junctions superconding devices which could one day replace the existing semi-conductor based computers.

The PhD Manuscript is composed of a short introduction commenting the results of four articles that are given in the appendix. The text, apart from the nice discussion on Josephson superconducting devices, is very general and does not provide many explanations on the methods and techniques used in the articles; I will therefore comment these four articles.

1 Modeling kink dynamics in the sineGordon model with position dependent dispersive term

In this study Gatlik compares four different collective variable methods to describe the following perturbed sine-Gordon equation

$$\phi_{tt} - \partial_x(F(x)\phi_x) + \sin\phi = 0.$$

The justification of the model seems complicated, an easier justification is a 1D averaging of a 2D sine-Gordon (Josephson junction) with a variable transverse width, see for example [bcs96] and the reference to Pagano therein. Another way would have been to derive a 1D model from [cgscc03] which considered the 2D curved case.

The study focuses on $F(x)$ being a "bump" like function. The main effect of the perturbation of the kink comes from the positive gradient of $F(x)$ so a more detailed study of the interface region would be useful. Maybe just analyze first $F(x) = \tanh(x/d)$. I expect that for d large the kink crosses and just slows down, this is the slow varying regime and probably can be described by soliton perturbation theory. What happens when d becomes comparable to the soliton width ? Another point is to look at what happens when $F(x)$ decreases. We would have a potential well instead of a barrier. This could be useful for certain applications.

The four variational methods are an energy method, the soliton perturbation of McLaughlin and Scott, the projection on the zero mode ϕ_{x_0} and a projection on the energy density $\phi^2_{x_0}$. Interestingly, it is this fourth method that agrees best with the full numerics. This is a new and useful result.

As usual for collective variables, it is difficult to understand why they work and why not. Maybe soliton perturbation theory could help with this analysis. Gatlik provides a justification for the better agreement of the method based on the projection on the energy density. The equations for the third method are the Euler-Lagrange equations of a reduced average Lagrangian, see [1991]. It would be interesting to see if the energy density (fourth method) is also related to a reduced Lagrangian.

2 The impact of thermal noise on kink propagation through a heterogeneous system

The equation analyzed in this article is the driven damped inhomogeneous sine-Gordon

$$\phi_{tt} - \partial_x(F(x)\phi_x) + \sin\phi = -a\phi_t + \Gamma.$$

where $\Gamma(t)$ is a thermal noise. The article is difficult to read and understand for non specialists. It does present a nice derivation of a Fokker-Planck equation from the collective variable model based on energy density.

The solutions of the equations derived in Appendix B of article are in excellent with the stochastic PDE solutions, this is shown in Figs 3-6. These results are interesting and can be related to experiments done at low temperature. Also, it would have been nice to give more details on the machinery leading to the Fokker-Planck equation.

3 Kink-inhomogeneity interaction in the sine-Gordon model

In this article Gatlik considers

$$\phi_{tt} - \partial_x(F(x)\phi_x) + \sin\phi = -a\phi_t + \Gamma,$$

where $F(x) = 1 + g(x)$ and examines the dynamics of the kink inside the perturbed domain ie the support of g.

The interesting case corresponds to the current Γ "pushing" the kink into the domain. Then we get a stable static kink ϕ_0

$$-\partial_x(F\phi_{0_x}) + sin\phi_0 = -Gamma.$$

Gatlik then writes $\phi(x,t) = \phi_0(x) + \psi(x,t)$. The perturbation ψ satisfies

$$\psi_{tt} - \partial_x(F\psi_x) + cos(\phi_0)\psi = 0$$

which can be solved using separation of variables

$$\psi = \exp(iwt)v(x)$$

to lead to the following spectral problem

$$-\partial_x(Fv_x) + cos(\phi_0)v = w^2 v.$$

Following the procedure detailed above, Gatlik derives the eigenfrequencies describing small oscillations around ϕ_0. These are negative for $\Gamma = 0$ so that ϕ_0 is unstable as expected.

Gatlik then proceeds to a write a collective variable description of the field ϕ. He uses three methods
1. Conservative Lagrangian $a = \Gamma = 0$
2. Projection on zero mode

3. Non Conservative Lagrangian : this method is recent and not well known. It would have deserved a more detailed description, both in the article and the manuscript.

Interestingly methods 2 and 3 yield the same ODE for the center of the kink x_0. Why is that ? then why include method 3 ? For a two degrees of freedom description, center and width, x_0, γ methods 2 and 3 differ. Why is that ? The non conservative Lagrangian method performs better, why is that ?

Finally note that the material in the Appendix: equivalence of the first two approaches for one degree of freedom reduced models is known , see [1991].

4 An effective description of the impact of inhomogeneities on the movement of the kink front in 2+1 dimensions,

The authors analyze the following PDE

$$\phi_{tt} - \partial_x((1 + g(x, y))\phi_x) - \phi_{yy} + \sin\phi = -a\phi_t + \Gamma.$$

The model seems a bit ad hoc; it would be interesting to have a physical context for it.

Using the nonconservative Lagrangian machinery, they derive the following equation for the center of mass

$$X_{tt} + aX_t - X_{yy} + \epsilon(q(y)/2)V(X) = \Gamma\pi/4$$

Could this equation have been derived using simpler arguments ? It seems like it.

Three different situations were analyzed
A. Kink propagation in the absence of inhomogeneities
B. Propagation of the front in the presence of an x-axis-directed inhomogeneity
C. Kink propagation for inhomogeneities dependent on both variables
In these three situations the full PDE and the reduced PDE for X are in excellent agreement.

To summarize, Jacek Gatlik has presented a very complete analysis of this perturbed sine-Gordon equation both in 1D and 2D. He also examined the influence of thermal noise on the solutions. The agreement observed between the PDE solutions and the solutions of the collective variable equations is remarkable. As is usual for collective variable studies, this agreement is not guaranteed, it would be interesting to justify it, maybe using soliton perturbation theory. It would be useful to understand in depth the effect of the gradient of F on the kink. Finally, the connection to experiments should be emphasized. In the continuation of the discussion of Josephson junctions of the introduction, it would have been interesting to relate the results obtained to different experimental situations.

Despite these four articles being collective work, it is clear that a very large part was conducted by Jacek Gatlik. In particular, he wrote all the codes to solve the PDEs and the collective variable equations.

To conclude, the quality of the results obtained, the amount of work done, and the number of different techniques used, both numerical and analytical, clearly demonstrate that Jacek Gatlik is an accomplished researcher. I request that M. Jacek Gatlik be admitted to the public defense of his PhD.

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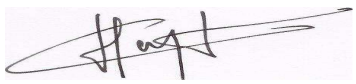

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