Review of the doctoral dissertation

Title: **Asymptotic invariants of configurations of points determined by complex reflection groups**
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Summary:

The thesis under examination studies some homological and geometric invariants of certain configuration of points in $\mathbb{P}^2$ and $\mathbb{P}^3$ defined via root systems.

The interest in configurations of points in $\mathbb{P}^n$ defined by root systems gain lot of attention after the work [3], where the authors proved by direct computation that some sets of points coming from root systems of reflection groups give examples of unexpected hypersurfaces and curves.

Furthermore, in the appendix of [2], two of these configurations in $\mathbb{P}^3$, named D_4 and F_4, are reported to have unexpected cones in specific degrees making their general projection a complete intersection. (Projective varieties having such property are called *geproci*; the nomenclature was introduced in [4].) The easiest examples of geproci sets are grids in $\mathbb{P}^3$, which are well studied in [2] with respect the point of view of unexpected cones.

A deeper study of the geometry and the combinatoric of the D_4 and F_4 configurations can be find in [1], which also contains a larger analysis of geproci sets.

Content of the thesis:

The main results of the thesis under examination are framed in this line of research. Specifically, in order to compute certain asymptotic invariants of sets of points derived from root systems of real reflection groups, direct computation alone is not sufficient, and additional geometric information is needed. In particular, in Section 4 of this thesis, the examination of point-plane and point-line incidences of X allows the author to compute the Waldschmidt constant, $\hat{\alpha}(I_X)$, for the configurations of points X defined by the root systems D_4, B_4 and F_4 (Theorem 4.7, Theorem 4.9 and Theorem 4.15). The proofs are of interest since they involve the intrinsic symmetry of such sets.

Other invariants, related to the Waldschmidt constant of $\hat{\alpha}(I_X)$ for $X = D_4, B_4, F_4$, are either computed or estimated in Section 4. For instance, the knowledge of $\hat{\alpha}(D_4)$ and $\hat{\alpha}(F_4)$ allows the author to compute (or at least bound) the resurgence of certain configurations contained in F_4 .

It is also of interest the computation of $\hat{\alpha}(H_4)$, which remains an intriguing open problem, but it is shown to be smaller than the trivial bound (Theorem 4.24).

Chapter 5 includes the results of [5] where the authors complete the computation initiated in [3] about the unexpected cones of the root system H_4. In [3] it was computed that H_4 admits an unexpected cone of degree 6 with vertex at a general points. Here the computation is done for the cones of degree 10. Since a cone of degree 10 with vertex at a general point can be chosen to be irreducible, then the H_4 configuration is a (6,10)-geproci (notation introduced in [1] to underline that the general projection is a complete intersection of two curves of degree 6 and 10). The appendix includes codes with singular for computing the examples presented in the paper and tables of incidence of the mentioned configurations. The first three sections introduce the notation and the background, give an overview of the problems and include some useful examples.

Originality and contribution to the field:

The thesis contains an original contribution to the knowledge of the geometry and the asymptotic invariants of sets of points defined by root systems of reflection groups.

The results in Sections 4 are an original contribution to the knowledge of the asymptotic invariants such as Waldschmidt constant and resurgence of special configurations of points. They fit into within a current research area of widespread interest. The approach to the computation of these invariants is standard but it involves a deep study of the geometry and combinatoric of these configurations of points defined by root systems of reflection groups. Section 5 gives a contribution on the theory of geproci sets providing an example of a set of points not contained in a curve of degree b in $\mathbb{P}^3$ whose general projection is a complete intersection of type (a, b), $b \geq a$, (examples of this kind, called non-trivial non-halfgrid geproci sets, seems to be very evasive).

The structure of the thesis is coherent, the methodology seems appropriate for the research question, and it is applied correctly.

In my opinion the presented thesis fulfills all legal and traditional requirements and I request that the submitted dissertation be admitted for defense.

Comments and suggestions.

Some work could be done to improve the exposition of the focal points. Here are some suggestions

1. Proof of Theorem 4.7. The first line of the proof, *computing the value 2*, should be rephrased, maybe, explaining that we are going to compute such value by contradiction. I don't see why S, at page 45, should be *symmetric*, however this does not affect the proof since it is actually proved that S must contain all the planes. *Since this goes forever* should be something like *Hence we can apply again the same argument as above and so [...].*

2. The main result of the work, Theorem 4.15, has a very clean statement followed by a proof that should be written in a more clean way. I suggest to move the geometrical analysis of the F_4 configuration in remarks or lemmas before the statement and leave in the proof only the computation of $\hat{\alpha}$. Also it would be better to explicitly say that the proof shows by induction that

   For $d \geq 2$, no surface S of degree $24 \cdot 10^d - 1$ vanishing at the 24 points of the F_4 configuration with multiplicity $9 \cdot 10^d$.

   where the *special case* is indeed the basis of the induction $d = 2$ and the *general case* is the case $d > 2$.

The candidate shows a critical understanding of the relevant literature and the ability to critique their own work. However, some remarks from [1] are mentioned but the work does not appear in the bibliography. I suggest, for example, to mention at page 46 that the the harmonic property of four collinear points in the F_4 configuration (and then B_4) is in Remark 1.4. in [1]; at page 49 observe that F_4 is a $(4, 4)$-grid plus two 8 points equidistribuited in two skew lines is in Example 4.4. in [1]; at page 55 that the 32 lines containing exactly three points of F_4 are given in table (6.3) in [1]; at page 79 that the notation of (a, b)-geproci for sets for points in $\mathbb{P}^3$ (used in Theorem 5.6 but not defined in this terms in the thesis) was introduced in [1].

The thesis is mostly well-structured but the language could be improved in some parts. There are several mistakes in English that make difficult the reading of certain paragraphs. For instance:

- page 13 line 5: *later* should be *latter*; (also at page 45)
- page 34 line 4: *revoke* should be *invoke*(?); (also at page 54 and 63)
- page 49 line -1 *much* should be *match*;
- page 57 line 11, *before* should be *below.*
- page 79 *flat* is not as common as *plane.*

In the following list there are some typos and other minor comments improve the exposition:

- Remark 1.6. X_1 and X_2 are not defined.
- Definition 1.17 *its ideal is* replace with *if its ideal is*
- In Definition 1.27 *calc* instead of $\mathcal{C}$. Also, in the first item correct $I(P, \mathcal{C} \cap \mathcal{D})0$ should be $I(P, \mathcal{C} \cap \mathcal{D})$. In item 5 it is not clear what *There is* means at the beginning.
- page 26 line above Remark 2.11 seems incomplete.
- page 44 the 12 given points came exactly from the definition of D_4. A change of variables is not needed.
- page 45. The set of points in Section 4.3 is called $Z(B_3)$ in the definition and just B_3 later. The notation should be unified. (See also $Z(F_4)$ at page 48.)
- Page 46 *cross-ration* should be cross-ratio.
- Page 47 line 4 *to increase* should be *to do not decrease.*
- Page 47, maybe is better to do not call the polynomial P, since P is denoting points in the same page.
- Page 52 line -6, the definition of $\hat{\alpha}$ was given for ideals. Here is written $\hat{\alpha}(Z)$ instead of $\hat{\alpha}(I_{Z_9})$. Possibly it can be added to the definition that when I is the ideal defining the set of points Z then we write $\hat{\alpha}(Z)$ instead of $\hat{\alpha}(I_Z)$.
- page 57 line -4, *trace of a surface* in my opinion is misleading;
- page 58 line -1, *Now, it turns out, that we in the position* could be *Now we are in the position*
- page 59 line -3, the notation for ideals defining set of points should be uniform. Here is $I(F_4)$, in other parts of the thesis is I_Z.
- page 75 line 3 *proof* should be *prove*
- page 78 the example in figure 5.1 is not an half-grid since the general projection of 6 points is not contained in a conic as required in the previous definition.

References

[1] L. Chiantini, Ł. Farnik, G. Favacchio, B. Harbourne, J. Migliore, T. Szemberg, and J. Szpond. *Configurations of points in projective space and their projections.* arXiv:2209.04820 (2022).

[2] L. Chiantini and J. Migliore. *Sets of points which project to complete intersections, and unexpected cones.* Transactions of the American Mathematical Society 374, no. 4 (2021): 2581-2607.

[3] B. Harbourne, J. Migliore, U. Nagel and Z. Teitler. *Unexpected hypersurfaces and where to find them.* Michigan Mathematical Journal 70, no. 2 (2021): 301-339.

[4] P. Pokora, T. Szemberg and J. Szpond. *Unexpected properties of the Klein configuration of 60 points in P3.* Michigan Mathematical Journal 1, no. 1 (2023): 1-17.

[5] Wiśniewska, Paulina, and Maciej Zieba. *Generic projections of the H4 configuration of points.* Advances in Applied Mathematics 142 (2023): 102432.